Adaptive Methods

High Resolution Recovery of Piecewise Smooth Data from its Spectral Information

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Outline

- Review of Fourier approximations for smooth(periodic) functions.

- Piecewise smooth functions - practical computational data.

- Adaptive Mollifiers for the high order resolution of Gibbs' Phenomena.

  - error analysis $\rightarrow$ exponential accuracy for piecewise analytic, numerics

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- Piecewise smooth functions - practical computational data.
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  - error analysis $\rightarrow$ exponential accuracy for piecewise analytic, numerics
- Adaptive Filters
  - overview of error analysis, numerics

Global(Periodic) Regularity and High Resolution

Spectral Convergence Rate, C^s

$$|S_N f(x) - f(x)| \leq Const \|f\|_{C^s} \cdot \frac{1}{N^{s-1}} \quad \forall s$$

⊙ $\|f\|_{C^s} := \max_{[-\pi,\pi]} |f^{(s)}|$

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Exponential Convergence Rate, Analytic

$$|S_N f(x) - f(x)| \leq Const_\eta \cdot N e^{-N\eta} \quad \Leftrightarrow \quad \|f\|_{C^s} \leq Const \cdot \frac{s!}{\eta^s}$$

Behavior of $f(\cdot)$ off the real axis determines η.

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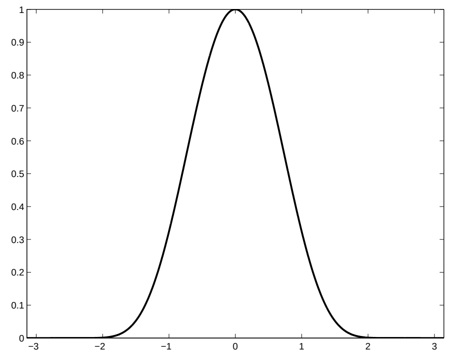
Behavior of $f(\cdot)$ off the real axis determines η.

- Convergence rate as fast as *Global* smoothness permits.
- What about $f \in C^\infty$ non-analytic? Gevrey regularity.

What about Spectral Convergence for C^∞?

$$\rho(x) = \begin{cases} \exp\left(\frac{(\pi x)^2}{x^2-\pi^2}\right) & |x| < \pi \\ 0 & |x| \geq \pi. \end{cases}$$



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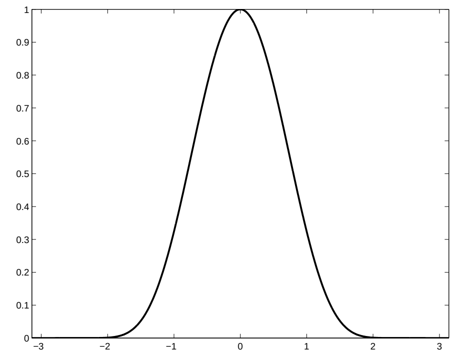

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$\rho(\cdot)$ is *Gevrey regular* of order 2, $\rho^{(s)} \sim (s!)^2$,

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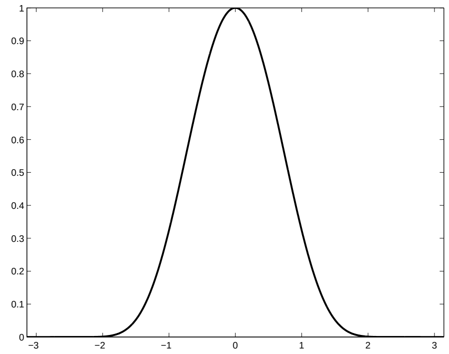

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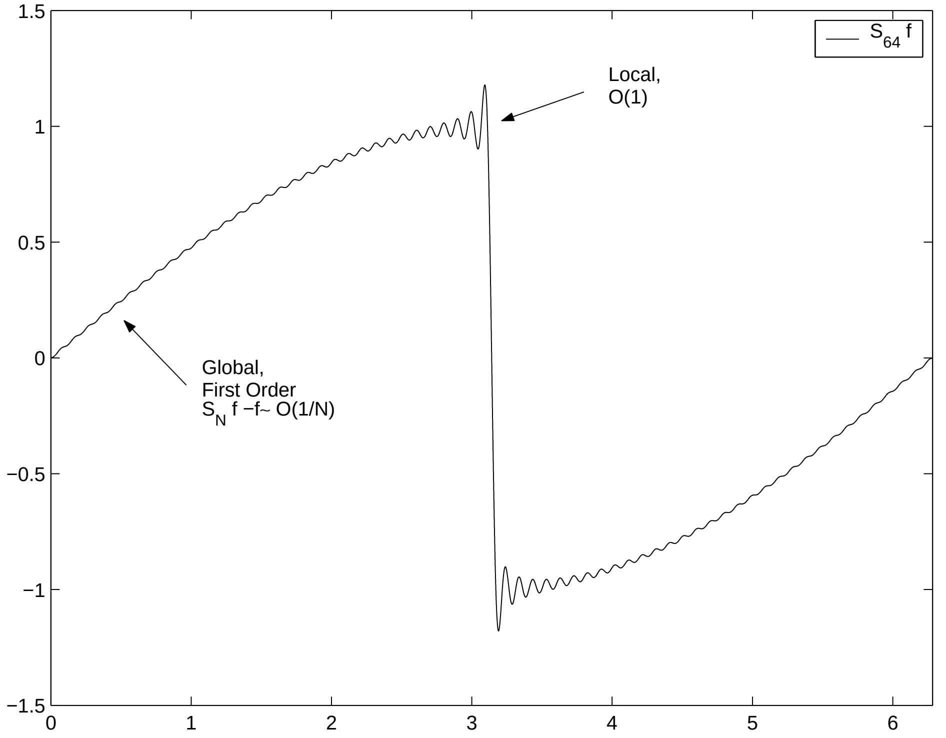
Fractional power exponential convergence.

$$|S_N\psi - \psi| \leq Const_\eta \cdot N^{\alpha/2}e^{-\alpha(\eta N)^{1/\alpha}} \quad \Leftrightarrow \quad \|\psi\|_{G^\alpha} \leq Const \cdot \frac{(s!)^\alpha}{\eta^s}$$

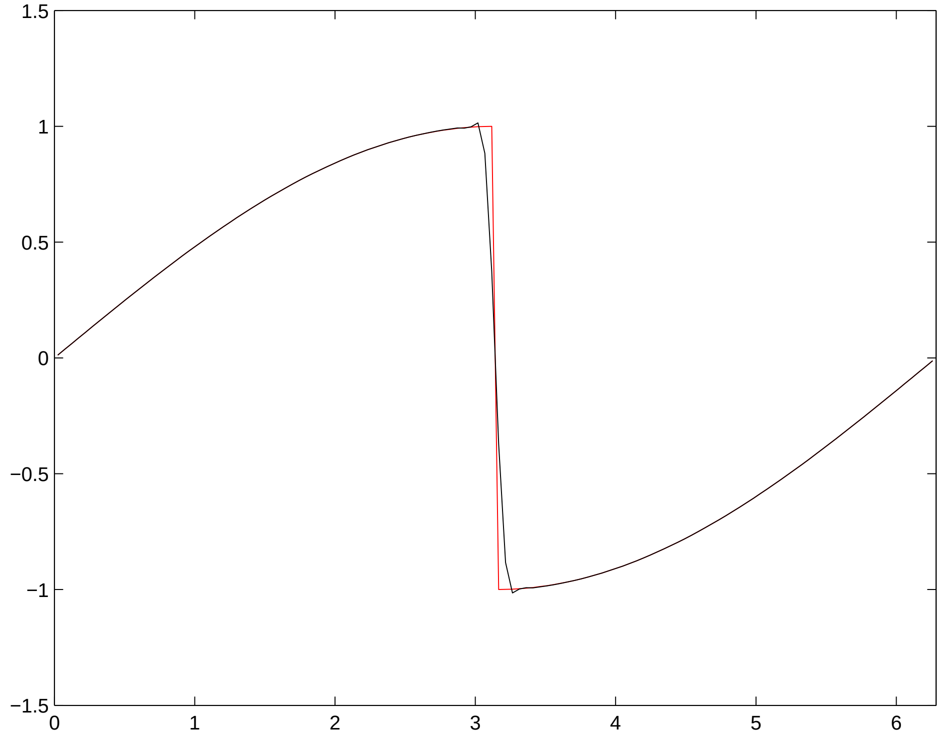
Gibbs' Phenomena and Filtered Reconstruction

$$f(x) = \begin{cases} \sin(x/2) & x \in [0, \pi) \\ -\sin(x/2) & x \in [\pi, 2\pi) \end{cases}$$



$S_{64} f(x)$

$\Rightarrow$



$S_{64}^{\sigma} f(x)$

- High order reconstruction with Gibbs' Phenomena removed.

Methods for discontinuous data

Local Smoothness, fixed polynomial order

$$\left.\begin{array}{l}\bullet\ \text{Splines}\\ \bullet\ \text{Wavelets}\\ \bullet\ \text{WENO}\end{array}\right\} P_N f - f \sim O\left(\frac{1}{N^r}\right) \quad \text{fixed r}$$

Semi-Global Smoothness, spectral accuracy

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Filtering and Mollification are essentially interchangeable,

$$\sum_{|k|\le N} \sigma\left(\frac{|k|}{N}\right) \hat{f}_k e^{ikx} \quad \Leftrightarrow \quad \phi * S_N f(x)$$

Dual Space   Physical Space

⊙ When to filter in dual space or mollify in physical space?

⊙ Computationally more efficient to stay is the space of given data.

Canonical Polynomial Order Mollifiers

- *One-parameter* compactly supported, $(-\pi, \pi)$, functions.

$$\psi_\theta(x) := \frac{1}{\theta}\psi\left(\frac{x}{\theta}\right), \quad -\pi\theta \le x \le \pi\theta$$

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- Convergence order as fixed number of exactly vanishing moments

$$\int_{-d}^{d} x^j \psi_\theta(x)dx = \delta_{j,0} \qquad j = 0, 1, 2, \ldots, r-1.$$

- Local $\quad |f - f * \psi_\theta| \le \frac{\theta^r}{\pi^r(r+1)!}\|f^{(r)}\|_{L^\infty(x-\pi\theta, x+\pi\theta)}$

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- Error decreases at *fixed polynomial order*(θ^r), $\theta \downarrow 0$.

- Recover from $S_N f(\cdot)$ requires similar order regularity, $\psi \in C^r$.

- Inherent small scale introduced by projection(sampling), $h \sim 1/N$.

Adaptive Mollifier of Tadmor(Gottlieb) & Tanner

The *two parameter* mollifier is given by:

$$\psi_{p,\theta}(x) := \frac{1}{\theta}\rho\left(\frac{x}{\theta}\right) D_p\left(\frac{x}{\theta}\right)$$

⊙ $\rho(\cdot)$, our G_0^2 localizer $\rho(x) = \begin{cases} \exp\left(\frac{(\pi x)^2}{x^2-\pi^2}\right) & |x| < \pi \\ 0 & |x| \geq \pi. \end{cases}$

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- Cancellation: $\psi_{p,\theta}$ possesses p *near vanishing moments*.

$$\int_{-\pi\theta}^{\pi\theta} x^j \psi_{p,\theta}(y)dy = \delta_{j0} + C_j \cdot p^{-(j-1)}, \ \ \forall j \leq p$$

- Unlike traditional mollifiers, dilation parameter as large as allowable

Error Analysis of Adaptive Mollifiers

Pointvalue recovery from a spectral projection, $S_N f(\cdot)$, or equidistant sampling $I_N f(\cdot)$.

Error composed of two terms:

$$\begin{aligned} \text{Error} &:= \psi_{p,\theta} * S_N f(x) - f(x) \\ &\equiv (f * \psi_{p,\theta} - f) + (S_N f - f) * (\psi_{p,\theta} - S_N \psi_{p,\theta}) \end{aligned}$$

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  - Controlled by number of near vanishing moments
- Truncation $:= (S_N f - f) * (\psi_{p,\theta} - S_N \psi_{p,\theta})$
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Error Analysis:

- The optimal number of near vanishing moments, p.
- Justify selection of dilation parameter, $\theta\pi := d(x)$.

Revisited: Error=Regularization+Truncation

Error Analysis I, Regularization Error

- $f(\cdot)$ analytic in $(x-d(x), x+d(x))$ where $d(x)$ distance to discontinuity, and $\rho \in G^2$, therefore $g_x(y) := f(x+y)\rho_{d(x)}(y) - f(x)$ is Gevrey order 2.

$$
\begin{aligned}
|\text{Regularization}| &:= |f * \psi_{p,\theta} - f| \\
&= |S_p g_x(y) - g_x(y)|_{\mathrm{y=0}} \\
&\leq C_\rho \cdot p \cdot e^{-2\sqrt{p \cdot \eta_\rho}}
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• Hidden dependence on dilation, θ, in η_ρ.

• Dilation parameter, θ, as large as possible such that $f(\cdot)$ analytic in $(x-\theta\pi, x+\theta\pi) \Rightarrow \theta\pi := d(x)$.

• Non-Linear Adaptive Mollification, $\theta(x) := d(x)/\pi$.

⊙ Symmetric reconstructions *must* sacrifice accuracy as approaching edges.

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Error Analysis II, Truncation error

- Smoothness reflected as dual space localization, $(\psi - S_N\psi)$, truncation.
- Dirichlet Kernel analytic and $\rho \in G^2 \Rightarrow\ \psi_{p,\theta(x)}$ Gevrey order 2,

$$\|\psi_{p,\theta(x)}\|_{C^s} \leq C_\rho \ \cdot\ s \cdot \left(\frac{s^2}{e^2\eta\theta(x)}\right)^s e^{p\eta/s} \qquad s = 1, 2, \ldots$$

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$$\begin{aligned}
|\text{Truncation}| &:= \|(S_N f - f) * (\psi_{p,\theta} - S_N\psi_{p,\theta})\|_{L^\infty} \\
&\le Const \cdot \left(\frac{1}{N}\right)^{s-1} \|\psi_{p,\theta(x)}\|_{C^s}, \ \ \forall s \\
&\le Const \cdot N \left(\frac{s^2}{e^2\eta\theta(x)N}\right)^s e^{p\eta/s}, \quad \forall s.
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$\theta(x)$ in denominator dictates $\Rightarrow \theta(x)$ as large as possible.

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Remaining:

- Equilibration of Regularization and Truncation Error.
- The optimal number of near vanishing moments, p.

Error Analysis III, Determination of parameter, p

- Truncation minimized over s when:

$$\log\left(\frac{s_{min}^2\pi}{\eta N d(x)}\right) = \frac{p\eta_c}{s_{min}^2} \quad \Rightarrow \; s_{min} \sim \sqrt{\eta \cdot N d(x)}$$

Incorporating this relationship for s_{min} yields

$$p_{min} = \frac{s_{min}^2}{\eta} \cdot \left(\log \frac{s_{min}^2 \pi}{\eta N d(x)}\right) \sim N d(x)$$

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- Adaptivity: The optimal choice for the number of near vanishing moments, p, is given as a function of the distance to the nearest discontinuity!

$$p_{min} := k \cdot Nd(x)$$

k selected to balance Regularization and Truncation errors, $k = 0.5596$.

Near discontinuities, Normalization

- $O(1/N)$ neighborhood of discontinuity, $p \sim Nd(x) \approx 1$.

- *Error for vanishing moments is substantial.*

$$\int_{-d}^{d} y^j \psi_{p,d}(y) dy = S_p * (y^j \rho(y))\Big|_{x=0} \sim \delta_{j0} + Const_j \cdot p^{-(j-1)}$$

- Visible error near discontinuities (blurring).

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- To maintain at least first order accuracy, normalize to unit mass

$$\psi_{N,d(x)}^{norm} := \frac{\psi_{Nd(x)}}{\int_{-d}^{d} \psi_{Nd(x)} dx}.$$

- Can possess any fixed number of exactly vanishing moments.

- Polynomial convergence near edges and exponential accuracy away.

There are no free parameters

- Exponential Accuracy away from discontinuity, $d(x) > O(1/N)$.

$$|\psi_{Nd(x)} * S_N f(x) - f(x)| \leq C_\rho \cdot Nd(x) e^{-0.845\sqrt{\eta \cdot Nd(x)}}.$$

- Explicit reconstruction depending only on the projection order, N, and the discontinuity locations, $d(x)$.

- Computationally robust due to rapidly decaying localizer, $\rho(\cdot)$.

- Optimal number of near vanishing moments given adaptively by

$$p_{min}(N, x) := k \cdot Nd(x)$$

- Polynomial order accuracy in $O(1/N)$ neighborhood of discontinuities

Pseudospectral(Equidistant) Recovery

$$\widehat{f}_k := \int_{-\pi}^{\pi} f(x) e^{ikx} dx \quad \Rightarrow \quad \tilde{f}_k := \frac{\pi}{N} \sum_{\nu=-N}^{N-1} f(y_\nu) e^{iky_\nu} \quad y_\nu := \frac{\pi}{N}\nu$$

Exponential accuracy the same order as spectral projection:

$$\left| \frac{\pi}{N} \sum_{\nu=-N}^{N-1} \psi_{Nd(x)}(x - y_\nu) f(y_\nu) - f(x) \right| \leq Const_c \cdot (Nd(x))^2 e^{-\sqrt{\eta \cdot Nd(x)}}$$

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- The pseudospectral coefficients are not needed; only samples $f(y_\nu)$.

- Robust exponentially accurate method to recover intermediate function values given an equidistant sampling of a piecewise smooth function.

- Accuracy proportional to number of cells to nearest discontinuity.

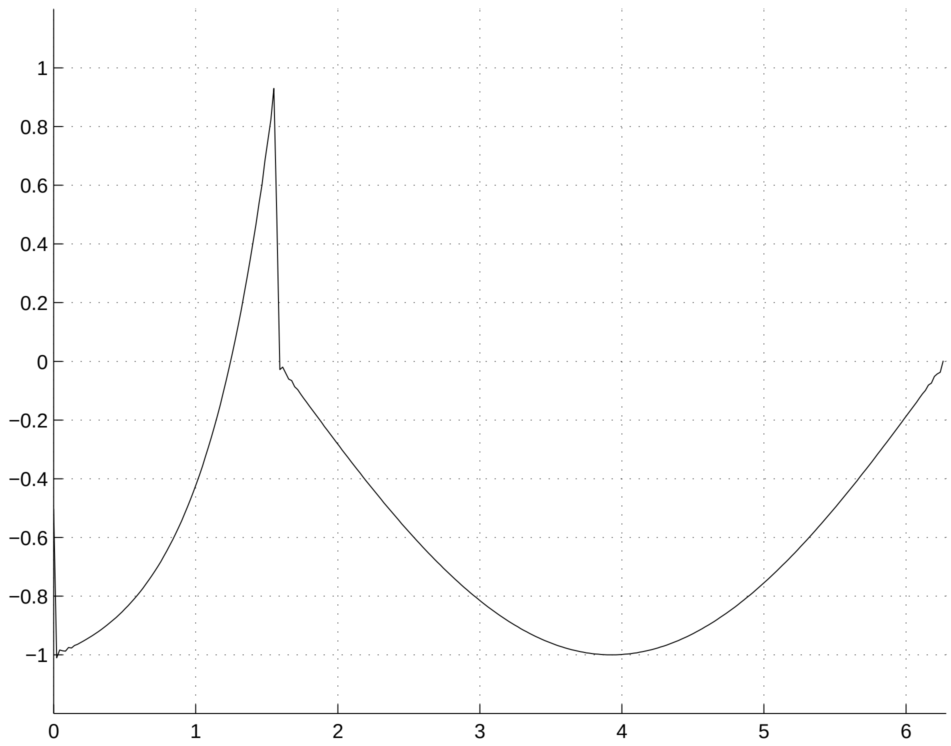
- Optimal order symmetric reconstruction, contrast with CWENO.

However, this is not Interpolation!

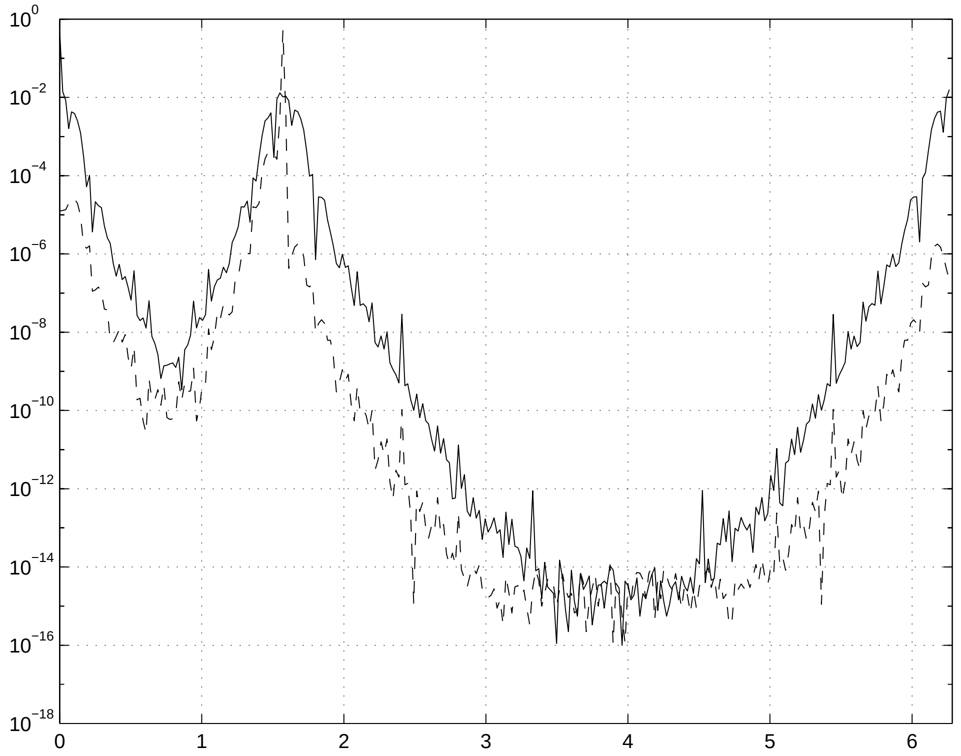
Reconstruction from Spectral Projection

$$f_2(x) = \begin{cases} (2e^{2x} - 1 - e^{\pi})/(e^{\pi} - 1) & x \in [0, \pi/2) \\ -\sin(2x/3 - \pi/3) & x \in [\pi/2, 2\pi) \end{cases}$$

Reconstruction $\psi * S_{128} f(\cdot)$



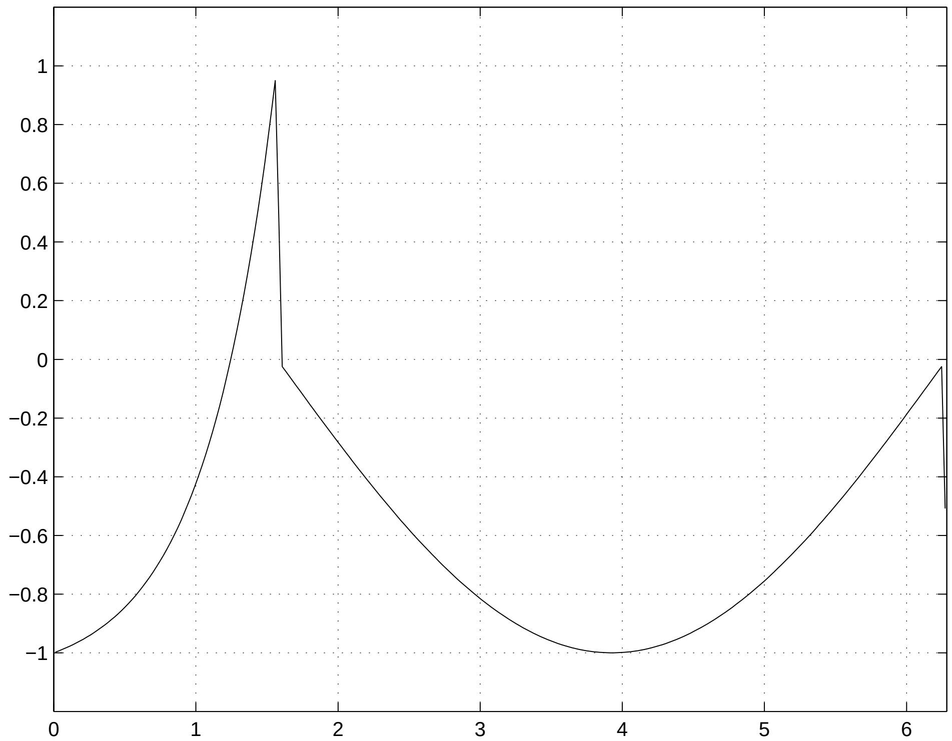
Regularization & Truncation errors



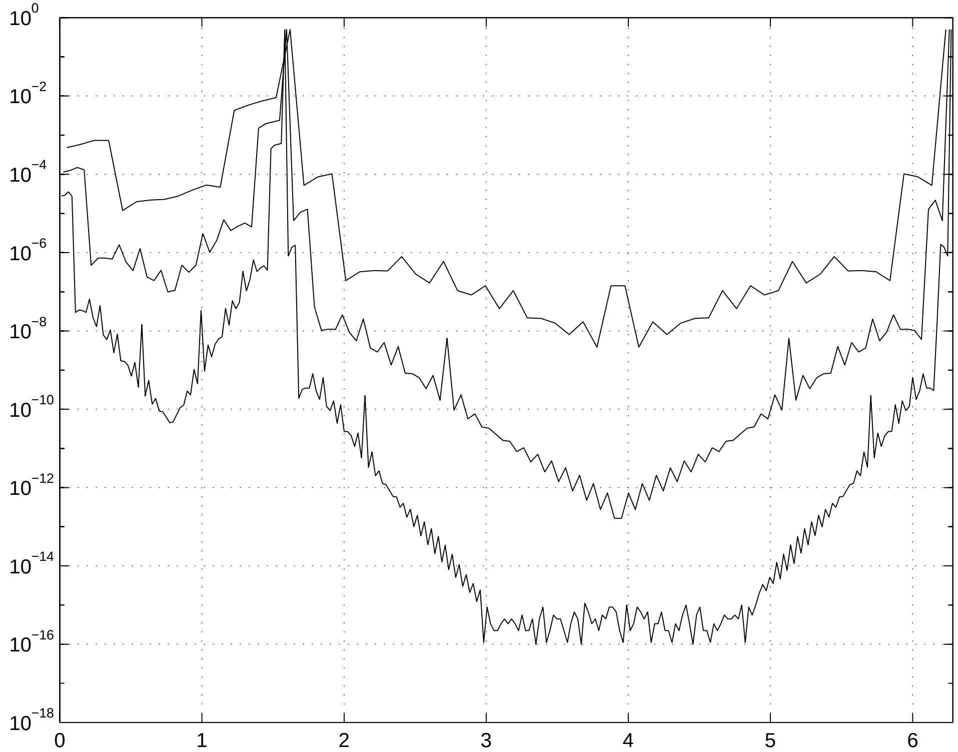
- Balanced decay rate of Regularization and Truncation errors
- Different regularity constants, steep gradient at $\frac{\pi}{2}^{-}$
- Accuracy sacrificed near discontinuities

Reconstruction from equidistant samples

Reconstruction, $\psi_{Nd(x)} * I_{128}f(\cdot)$



Log of error, $N = 32, 64, 128$

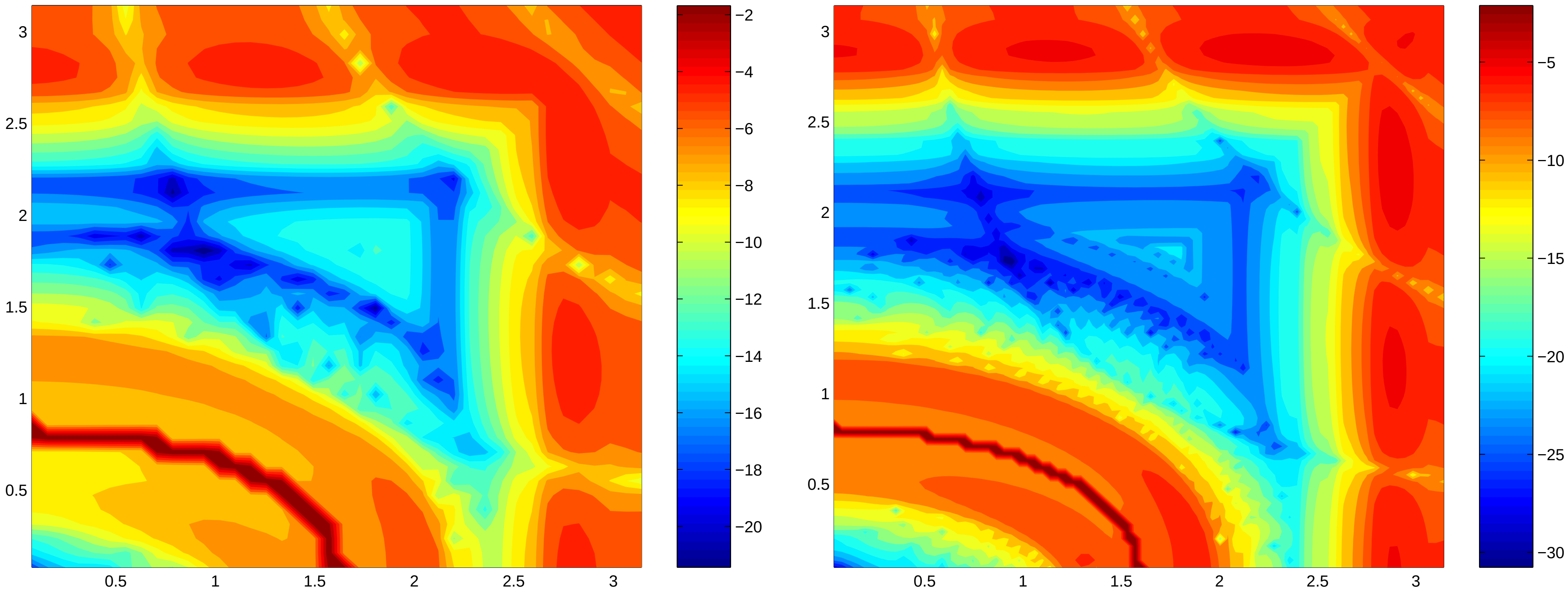


- Exponential convergence away from discontinuity
- Polynomial order accuracy near discontinuity, $d(x) = O(1/N)$
- Exact physical space localization, sharp resolution of discontinuities

2D Pseudospectral Example

$$f(x,y) = \begin{cases} \cos(xy) + 1 & 4x^2 + 16y^2 \leq \pi^2 \\ \cos(xy) & \text{else} \end{cases}$$

Error with $N = 40$ $\qquad$ $N = 80$



- Also treating boundaries as discontinuities, $x = \pm\pi, y = \pm\pi$.

Summary of Adaptive Mollifiers

• Recovers pointwise function values of piecewise smooth functions given either its spectral projection, or equidistant sampling.

• No parameters determined by the user, a "Black Box" method.

• *Computationally robust* and well suited for fast parallel computations.

• *Exponentially accurate* away from the discontinuity and fixed polynomial order convergence rate in the $O(1/N)$ neighborhood of edges.

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• Reconstruction errors a combination of

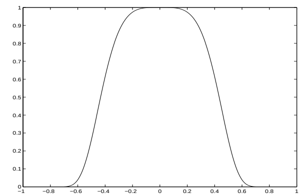
$$\left.\begin{array}{l} \text{– near vanishing moments, regularization} \\ \text{– physical space localization} \\ \text{– dual space localization, truncation} \end{array}\right\} \Rightarrow \text{Adaptive Filters}$$

Filters - Classical Polynomial Order

- Piecewise smooth functions, slowly decaying coefficients, $\hat{f}_k \leq O(k^{-1})$

- Filters increase convergence order by increasing coefficient decay rate

Filter properties $\Rightarrow \begin{cases} \text{- smoothness, } \sigma(\eta) \in C_0^q[-1,1] \\ \text{- accuracy, } \sigma^{(j)}(0) = \delta_{j0}, \quad j \leq q-1 \end{cases}$



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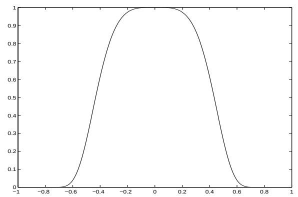

- Acts on function and its projection the same, dual space localization

$$f^\sigma(x) \equiv S_N f^\sigma(x) := \sum_{|k|\leq N} \sigma\left(\frac{|k|}{N}\right) \hat{f}_k e^{ikx}$$

- Error analysis through associated mollifier, acting in physical space

$$S_N f^\sigma(x) := f * \Phi(x) = \int_{-\pi}^{\pi} \Phi(y) f(x-y)\,dy \qquad \Phi(x) := \sum_{|k|\leq N} \sigma\left(\frac{k}{N}\right) e^{ikx}$$

Error Analysis - Sketch

- Accuracy condition implies a number of near vanishing moments

$$\sigma^{(j)}(0) \equiv \delta_{j,0} \quad \Rightarrow \quad \int_{-d}^{d} y^j \Phi(y)dy \sim \delta_{j,0} \qquad j = 0, 1, 2, \ldots, q-1.$$

- However, increased filter order, q, decreases localization

- Error composed of competing localization and accuracy errors,

$$|f(x) - f * \Phi(x)| \leq \left| \int_{d(x)<|y|\leq\pi} \Phi(y) g_x(y) dy \right| + \left| \int_{|y|\leq d(x)} \Phi(y) g_x(y) dy \right|$$

where, $g_x(y) := f(x) - f(x-y)$.

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- Fixed order filters fail to balance accuracy and localization errors

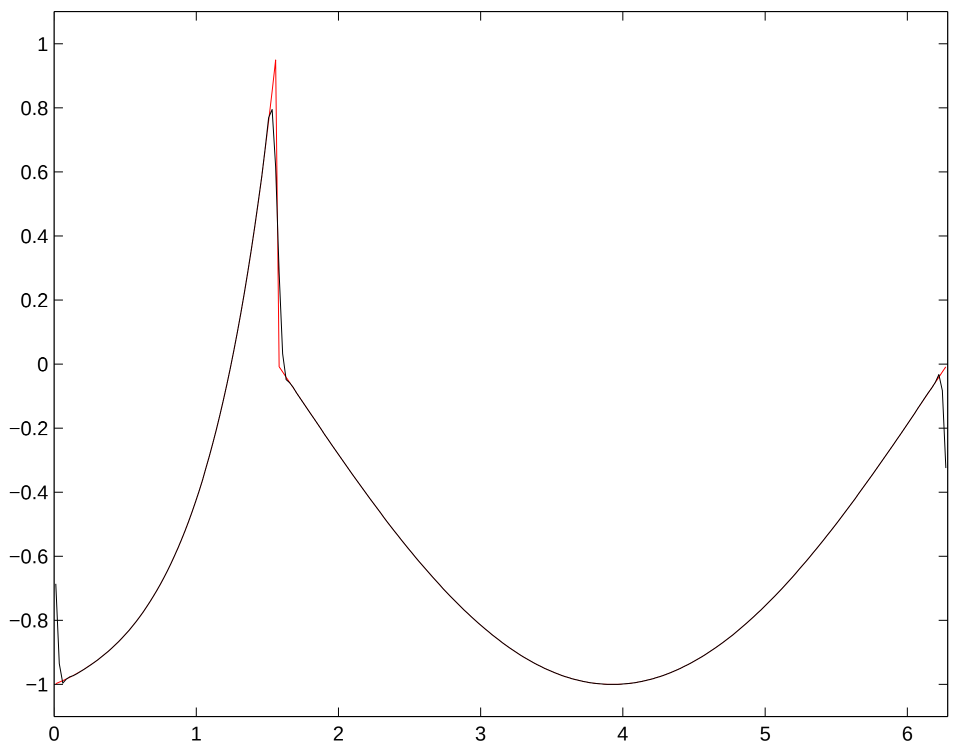
- Optimal filter order is spatially adaptive, balancing competing errors

$$q_{min} := (k \cdot Nd(x))^{1/\alpha} \qquad \sigma \in G_0^\alpha[-1,1]$$

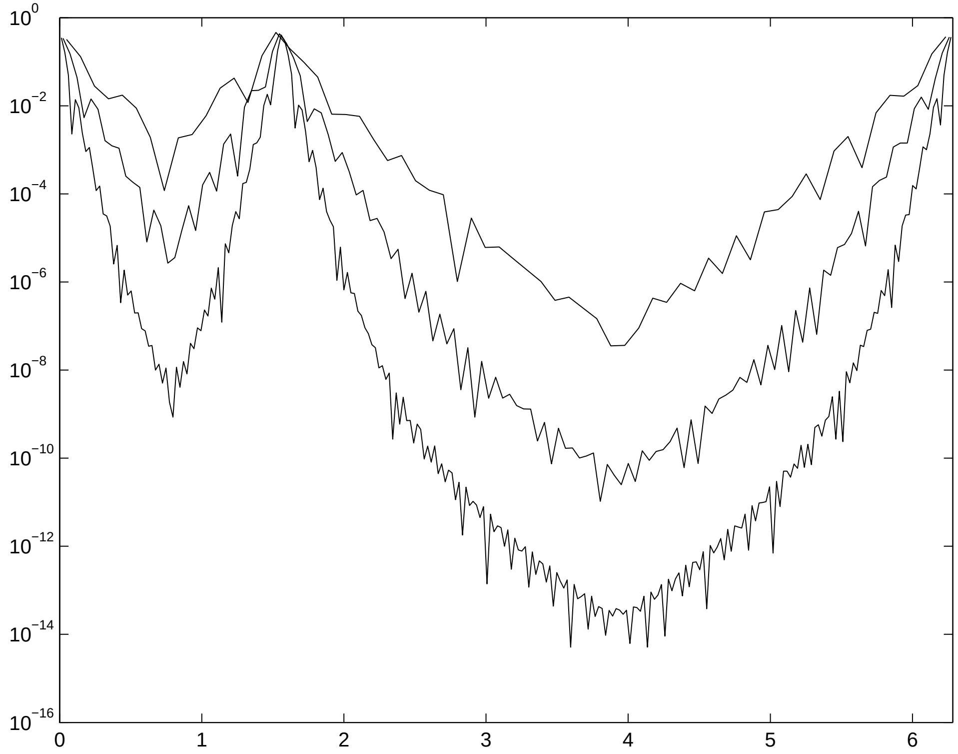
Reconstruction with Adaptive Filter

$$f_2(x) = \begin{cases} (2e^{2x} - 1 - e^{\pi})/(e^{\pi} - 1) & x \in [0, \pi/2) \\ -\sin(2x/3 - \pi/3) & x \in [\pi/2, 2\pi) \end{cases}$$

Reconstruction, $S_{128} f^{\sigma}(\cdot)$



$|f(x) - S_N f^{\sigma}(x)|$, $N = 32, 64, 128$



- Similar accuracy to Adaptive Mollifier
- Computationally fast when given spectral projection

Adaptive Mollifiers & Filters

A Powerful tool for manipulating a function's spectral projection or equidistant sampling.

Thank you

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